OPTIMUM CONTROL OF THE HEATING OF MASSIVE BODIES

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The general problem of controlling the heating of massive bodies in chamber furnaces is formulated, and optimum control solutions are given for two cases of heating of a plate with Newtonian heat transfer at its surfaces.

Let us assume that a body is being heated in a furnace, the body occupying a certain region D bounded by a surface G. Let us further assume that the temperature of the heating medium $u^*(t)$ does not depend on the space coordinates.

Then the heating process for the body can be expressed by the equation of thermal conduction

$$\frac{\partial Q}{\partial t} = a\left[\frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2}\right], \tag{1}$$

where $(x, y, z) \in D$, $t > 0$,
together with the initial condition

$$Q(x, y, z, 0) = Q_0(x, y, z) \tag{2}$$

and the following boundary condition of type III

$$\frac{\partial Q}{\partial \vec{n}} = \alpha_1\{[u^*(t)]^4 - [Q]^4\} + \alpha\{u^*(t) - Q\}, \tag{3}$$

where $Q = Q(x, y, z)$ and $(x, y, z) \in G$.

As the control function we shall take the time function $v(t)$, describing the position of the valve which opens to admit fuel to the working space of the furnace, and impose on this function the limitation

$$A_1^* \leqslant v(t) \leqslant A_2^*, \tag{4}$$

where A_2^* corresponds to the extreme position of the valve, at which a maximum quantity of fuel enters the furnace, while the other extreme position of the valve $v(t) = A_1^*$ corresponds to complete shutdown of fuel admission.

The relation between $u^*(t)$ and $v(t)$ must be assigned so as to introduce function $v(t)$ into boundary condition (3) of our problem. Using the language of automatic control theory, we need to find the transfer function of the system whose input is the control action $v(t)$, and whose output is the temperature of the heating medium. In the simplest case, the transfer function may be represented in the form of two elementary links connected in series: a delay link, taking into account the lag due to the finite rate of gas supply and the finite length of the pipeline between the control valve and the working space of the furnace, and an inertia link, taking into account the gradual rise of temperature in the furnace for a stepwise increase in fuel supply. The equation of this transfer function has the form

$$B\frac{du^*(t)}{dt} + u^*(t) = k_0 v(t - \tau). \tag{5}$$

To formulate the optimum heating problem, we must take into consideration a number of limits, the most important being the limits of the surface temperature, temperature gradients, and temperature drops in the body.

These may be written in the form:

$$Q(x, y, z, t) \leqslant A_3 \quad \text{when} \quad (x, y, z) \in G, \tag{6}$$

$$|\operatorname{grad} Q(x, y, z, t) \leqslant A_4 \quad \text{when} \quad (x, y, z) \in D, \tag{7}$$

$$\max|Q(x_1, y_1, z_1, t) - Q(x_2, y_2, z_2, t)| \leqslant A_5 \tag{8}$$

$$(x_1, y_1, z_1) \in D, \quad (x_2, y_2, z_2) \in D.$$

The quality of heating can be expressed as the following functional:

$$I_0 = \iiint_D [Q(x, y, z, T) - Q^*(x, y, z)]^2\, dxdydz.$$

The following formulations of the problem of optimum heating control are of practical interest.

1. For the conduction equation (1)-(3) and the coupling equation (5) choose a control action $v(t)$, $0 \le t \le T$, restrained by condition (4), such that the functional I_0 satisfies the condition $I_0 \le \delta$ (where δ is some assigned number $\delta > 0$ denoting the accuracy of approximation to the desired distribution) in the minimum possible time T, while at any instant $0 \le t \le T_0$ the limitations (6), (7), (8) must be observed.

2. For conduction equation (1)-(3), coupling equation (5), and a given heating time, the control action $v(t)$, $0 \le t \le T$, restrained by condition (4), must be such that the functional I_0 attains its minimum possible value at time $t = T$, while the limitations (6), (7), (8) are observed throughout the entire heating process.

Let us examine in detail the solution of the simpler problems concerning optimum heating of a one-dimensional plate of thickness 2S in a medium with temperature $u^*(t)$, subject to the limitation

$$A_1 \leqslant u^*(t) \leqslant A_2. \tag{9}$$

Bearing in mind that the duration of all the intermediate processes in the furnace is incommensurably less than the overall heating time, we shall neglect the inertia of the furnace and the lag in the pipeline and take as control action the temperature of the medium $u^*(t)$.

We shall write the conduction equation for the plate in criterial dimensionless form:

$$\frac{\partial q(l,\ \varphi)}{\partial \varphi} = \frac{\partial^2 q(l,\ \varphi)}{\partial l^2}, \tag{10}$$

$$q(l,\ 0) = -\nu, \tag{11}$$

$$\frac{\partial q(+1,\ \varphi)}{\partial l} = \mathrm{Bi}\,[u(\varphi) - q(+1,\ \varphi)], \tag{12}$$

$$-\frac{\partial q(-1,\ \varphi)}{\partial l} = \mathrm{Bi}\,[u(\varphi) - q(-1,\ \varphi)]. \tag{13}$$

The conversion formulas to dimensionless quantities are:

$$q(l,\ \varphi) = \frac{2[Q(l,\ \varphi) - Q^*]}{A_2 - A_1}, \tag{14}$$

$$u(\varphi) = \frac{2[u^*(\varphi) - Q^*]}{A_2 - A_1}, \tag{15}$$

$$\nu = \frac{2[Q^* - Q_0]}{A_2 - A_1}, \tag{16}$$

$$\varkappa = \frac{A_2 + A_1 - 2Q^*}{A_2 - A_1}. \tag{17}$$

We shall assume the initial and the required temperature distributions in the plate to be constant and equal respectively to Q_0 and Q^*.

By virtue of (9), (14), (15), (17) the dimensionless control action $u(\varphi)$ is now restrained by the condition

$$-(1 - \varkappa) \leqslant u(\varphi) \leqslant (1 + \varkappa). \tag{18}$$

We shall find the control action $u(\varphi)$, $0 \le \varphi \le \varphi_0$, restrained by condition (18), such that the equality

$$q(l,\ \varphi_0) \equiv 0, \qquad -1 \leqslant l \leqslant +1. \tag{19}$$

is fulfilled in the minimum time φ_0. The solution of equations (10)-(13) may be written in the form [3]

$$q(l,\ \varphi) = \sum_{k=1}^{\infty} A_k \cos \mu_k l \left\{ -\nu \exp\,[-\mu_k^2 \varphi] + \mu_k^2 \int_0^{\varphi} u(\xi) \times \right.$$
$$\left. \times \exp\,[-\mu_k^2(\varphi - \xi)]\, d\xi \right\}. \tag{20}$$

It is not hard to see that to satisfy (19) for minimum φ_0 it is necessary and sufficient that function $u(\varphi)$ satisfy an infinite number of equalities of the type

$$\frac{\nu}{\mu_k^2}\int_0^{\varphi_0} u(\varphi)\exp[\mu_k^2\varphi]\,d\varphi \qquad (k=1,\ 2,\ldots), \tag{21}$$

the time φ_0 then being the minimum possible. In mathematics this problem is called the infinite-dimensional problem of moments. It was shown in [4] that a function $u(\varphi)$, $0 \le \varphi \le \varphi_0$, satisfying (21) exists, assumes only its boundary values $(1 - \varkappa)$ and $(1 + \varkappa)$, and has an unbounded but denumerable number of change-over points on the intercept $[0, \varphi_0]$, the point φ_0 being a limiting change-over point. The form of this function is shown in Fig. 1.

Solving the problem for finite n, i.e., k = 1, 2, ..., n in (21), a finite-number problem of moments, we can satisfy (19) to any degree of accuracy specified in advance. Methods of solution of finite-number problems of moments are set out in detail in [2] and [5]. In practice, to obtain the best approximation to a required distribution, it is usually enough to put n = 2, 3, 4. Here, the greater the parameter Bi, the more slowly the terms of series (20) decrease, and the larger must be the number n.

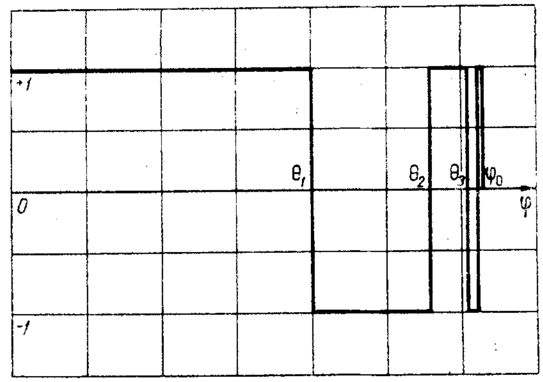

Fig. 1. Form of optimum control accurately realizing uniform temperature distribution in minimum time.

Let us consider a numerical example. Assume that it is required to heat a solid slab of thickness 2S = 0.4 m from an initial temperature $Q_0 = 20°$ to a temperature $Q^* = 960°$. Assume further that $a = \dfrac{0.03}{36\cdot 10^2}$ m²/sec, $\lambda = 30 \times 1.163$ w/m · degree, $\alpha = 225 \times 1.163$ w/m² · degree, and consequently, Bi = 1.5, $\varkappa = 0.6$, $\nu = 2.35$.

Using methods set out in [2], we find functions $u_2(\varphi)$, $u_3(\varphi)$, which transform to identity the first two or three of equations (21) in the minimum possible time. Returning then to dimensional quantities, we obtain

$$u_2^*(t) = \begin{cases} +1600°\,\text{C} & \text{when } 0 \leqslant t \leqslant 77 \text{ min}, \\ +\ 800°\,\text{C} & \text{when } 77 \text{ min} < t \leqslant 87; \end{cases}$$

$$u_3^*(t) = \begin{cases} +1600°\,\text{C} & \text{when } 0 \leqslant t \leqslant 78 \text{ min}, \\ +\ 800°\,\text{C} & \text{when } 78 \text{ min} < t \leqslant 88{,}5 \text{ min}, \\ +1600°\,\text{C} & \text{when } 88{,}5 \text{ min} < t \leqslant 89 \text{ min}. \end{cases}$$

The temperature distributions obtained in the plate for this control action are shown in Fig. 2.

Note that for engineering applications the difference between control actions $u_2^*(t)$ and $u_3^*(t)$ is insignificant. Indeed, a second switching of function $u_3^*(t)$ does not make sense, since its last interval of constancy is only 30 sec, and certainly comparable with the duration of the transient process in any real furnace.

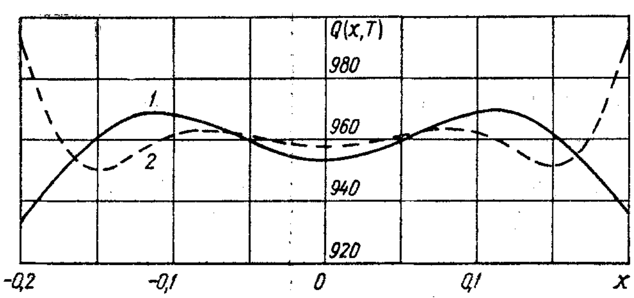

Fig. 2. Temperature distributions in plate: Bi = 1.5, $\varkappa$ = 0.6, ν = 2.35. 1 and 2 – for control actions $u_2^*(t)$ and $u_3^*(t)$.

We shall formulate the second plate problem as follows.

For equations (10)-(13) and a given time φ_0 we are required to find $u(\varphi)$, $0 \le \varphi \le \varphi_0$, restrained by condition (18), such that the functional $I_0 = \int_{-1}^{1} [q(l,\ \varphi_0,\ u(\varphi))]^2\,dl$ is minimized for this function.

By the method of the maximum principle [1], we can obtain the necessary condition, which the optimum control action $u(\varphi)$ must satisfy, in the form

$$u(\varphi) = \varkappa + \operatorname{sign}\left\{\sum_{k=1}^{\infty} B_k \exp[-\mu_k^2(\varphi_0-\varphi)]\left[\nu\exp(-\mu_k^2\varphi_0) - \mu_k^2\int_0^{\varphi_0} u(\varphi)\exp[-\mu_k^2(\varphi_0-\varphi)]\,d\varphi\right]\right\}, \tag{22}$$

where $B_k = \dfrac{2\mu_k \sin^2\mu_k}{\mu_k + \sin\mu_k\cos\mu_k}$.

So far, a general method of solution of equation (22) for any value φ_0 has not been found. However, in certain special cases this equation can be solved by a numerical method. For example, if the time φ_{03} (the minimum time for which problem (19) has a solution for n = 3) is fixed, then it is not hard to show that (22) does not admit solutions with a number of change-overs greater than two, for any $\varphi_0 \leq \varphi_{03}$, and, consequently, for this region the variation of time is easily sought. For example, a solution minimizing functional I_0 may be found by inspection.

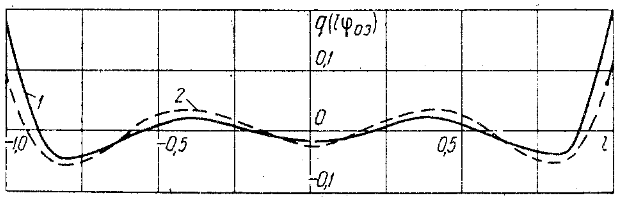

Fig. 3. Temperature distributions in plate: Bi = 3.0, x = 0.0, ν = 1.5; 1 — heating with control action $u_3(\varphi)$; 2 — optimum heating, in the sense of least mean square deviation, with control action $u_3^*(\varphi)$.

$$u_3(\varphi) = \begin{cases} +1 \text{ when } 0 \leq \varphi \leq 0.700, \\ -1 \text{ when } 0.700 < \varphi \leq 0.765, \\ +1 \text{ when } 0.765 < \varphi \leq 0.780; \end{cases}$$

$$u_3^*(\varphi) = \begin{cases} +1 \text{ when } 0 \leq \varphi \leq 0.702, \\ -1 \text{ when } 0.702 < \varphi \leq 0.767, \\ +1 \text{ when } 0.767 < \varphi \leq 0.780. \end{cases}$$

Fig. 3 gives the solution of the example Bi = 3.0, $\varkappa$ = = 0, ν = 1.5. Curve 1 is the temperature distribution obtained with control action $u_3(\varphi)$, $0 \leq \varphi \leq \varphi_{03}$, which solves problem (19) for n = 3. Curve 2 is the temperature distribution to which corresponds the control action $u_3^*(\varphi)$, $0 \leq \varphi \leq \varphi_{03}$, obtained by solving (22) with $\varphi_0 = \varphi_{03}$. Function $u_3^*(\varphi)$ is the optimum control action, since (22) does not admit other solutions reducing the value of the functional I_0. The values of I_0 for curves 1 and 2 are, respectively, 0.00282; 0.00251.

The results obtained indicate that in theory it is possible to solve the problems of optimum control of heating of a metal. The methods employed should be used to solve problems for bodies of more complex shape and also for more complex limitations.

NOTATION

Q(x, y, z, t) — temperature distribution in body; $u^*(t)$ — temperature of heating medium; a — thermal diffusivity; α — heat transfer coefficient; α_1 — radiant heat transfer coefficient; λ — thermal conductivity; **n**(x, y, z) — vector normal to surface G; τ — lag; B, k_0, A_3, A_4, A_5 — certain constants; T — total heating time; Bi = $\alpha S/\lambda$ — Biot number; $\varphi = \alpha t/S^2$ — dimensionless time (Fourier number Fo); Q^* — required temperature distribution in plate; Q_0 — initial temperature distribution in plate; $l = x/S$ — dimensionless thickness; A_2, A_1 — maximum and minimum possible furnace temperatures; $q(l, \varphi)$ — dimensionless temperature; μ_k — positive roots of the equation

$$\mu = \mathrm{Bi}\ \mathrm{ctg}\ \mu; \qquad A_k = \frac{2 \sin \mu_k}{\mu_k + \sin \mu_k \cos \mu_k}.$$

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